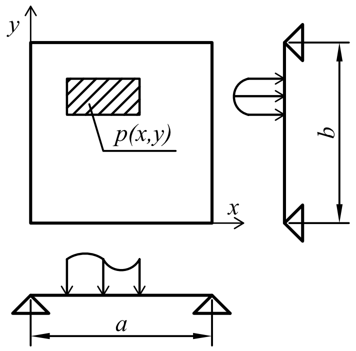
1.2 Razvijanje funkcije dve promenljive



Potrebno je zadato površinsko opterećenje *p(x,y)* u oblasti zadate pravougaone ploče razviti u dvostruki Furijeov red:

$$p(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}+B_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)$$

gde su periode $L_x=2a$, $L_y=2b$.

Površinsko opterećenje *p(x,y)* najčešće se prikazuje pomoću dvostrukih Furijeovih redova kao neparna funkcija kordinata *x* i *y* sa periodama $L_x=2a$ i $L_y=2b$, pa je:

$$p(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (*)$$

Da bi se odredili nepoznati koeficijenti A_{mn}, levu i desnu stranu jednačine (*) treba pomnožiti sa $\sin\frac{r\pi x}{a}\sin\frac{s\pi y}{b}$, gde su *r* i *s* celi brojevi, i zatim izvršiti integraciju u granicama od *0* do *a*, odnosno od *0* do *b*:

$$\int_0^a\int_0^b p(x,y)\sin\frac{r\pi x}{a}\sin\frac{s\pi y}{b}dxdy=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\int_0^a\int_0^b\sin\frac{m\pi x}{a}\sin\frac{r\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{s\pi y}{b}dxdy$$

$$\int_0^a\sin\frac{m\pi x}{a}\sin\frac{r\pi x}{a}dx=\begin{cases}\frac{a}{2}, m=r\\ 0, m\neq r\end{cases}, \quad \int_0^b\sin\frac{n\pi y}{b}\sin\frac{s\pi y}{b}dy=\begin{cases}\frac{b}{2}, n=s\\ 0, n\neq s\end{cases}$$

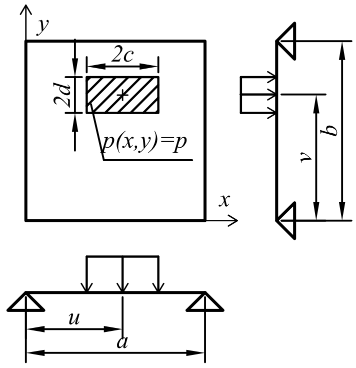
$$\int_b^a\int_0^b p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy=\frac{a}{2}\cdot\frac{b}{2}A_{mn}$$

$$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

Primer 1

Ravnomerno raspodeljeno optereénje unutar pravougaonika $2c\times 2d$ razviti u dvostruki Furijeov red.

$$p(x,y)=p=const \text{ za}\begin{cases}u-c<x<u+c\\ v-d<y<v+d\end{cases}$$



$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b p(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy = \frac{4}{ab}\int_{u-c}^{u+c}\int_{v-d}^{v+d} p\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy =$$

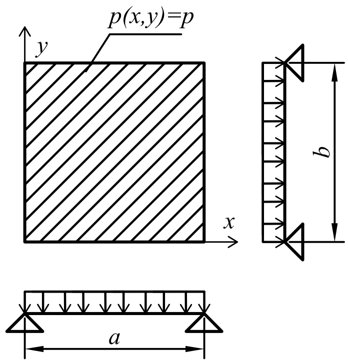
$$= \frac{4p}{ab}\int_{u-c}^{u+c}\sin\frac{m\pi x}{a}dx\int_{v-d}^{v+d}\sin\frac{n\pi y}{b}dy = \frac{4p}{ab}\cdot\left(-\frac{a}{m\pi}\right)\cos\frac{m\pi x}{a}\bigg|_{u-c}^{u+c}\cdot\left(-\frac{b}{n\pi}\right)\cos\frac{n\pi y}{b}\bigg|_{v-d}^{v+d} =$$

$$= \frac{16p}{\pi^2 mn}\sin\frac{m\pi u}{a}\sin\frac{m\pi c}{a}\sin\frac{n\pi v}{b}\sin\frac{n\pi d}{b} \qquad m,n=1,2,3,\ldots \qquad (**)$$

$$p(x,y) = \frac{16p}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\sin\frac{m\pi u}{a}\sin\frac{m\pi c}{a}\sin\frac{n\pi v}{b}\sin\frac{n\pi d}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Primer 2

Datu funkciju opeterećenja razviti u Furijeov red.



$$2c = a \Rightarrow c = \frac{a}{2}$$

$$2d = b \Rightarrow d = \frac{b}{2}$$

$$u = \frac{a}{2}, v = \frac{b}{2}$$

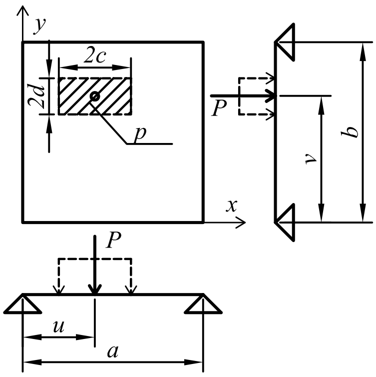
Iz prethodnog primera na osnovu izraza (**) dobija se

$$A_{mn}=\frac{16p}{\pi^2 mn}\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}=\begin{cases}\dfrac{16p}{\pi^2 mn}, & \text{za} \quad m,n=1,3,5,...\\ 0, & \text{za} \quad m,n=2,4,6,...\end{cases}$$

$$p(x,y)=\frac{16p}{\pi^2}\sum_m\sum_n\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad \text{za} \quad m,n=1,3,5,...,$$

Primer 3

Ploča opterećena koncentrisanom silom P u tački *(u,v)*.
Koncentrisana sila može se zameniti odgovarajućim ravnomerno raspodeljenim opterećenjem p na pravougaoniku dimenzija $2c\cdot 2d$. Razvoj ovakvog opetrećenja u trigonometrijski red dobija se određivanje granične vrednosti izraza (**), tj.



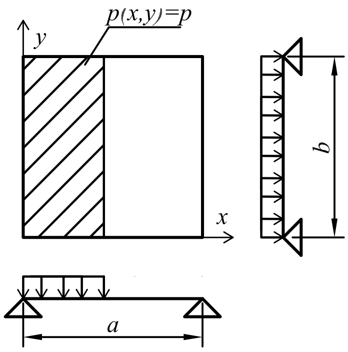
$$P=4cdp \qquad p=\frac{P}{4cd}$$

$$A_{mn}=\lim_{\substack{c\to 0\\ d\to 0}}\frac{16p}{\pi^2 mn}\sin\frac{m\pi u}{a}\sin\frac{m\pi c}{a}\sin\frac{n\pi v}{b}\sin\frac{n\pi d}{b}$$

$$A_{mn}=\lim_{\substack{c\to 0\\ d\to 0}}\frac{4\cdot 4p}{\pi^2 mn}\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}\frac{\sin\frac{m\pi c}{a}}{\frac{m\pi c}{a}}\cdot\frac{m\pi c}{a}\cdot\frac{\sin\frac{n\pi d}{b}}{\frac{n\pi d}{b}}\cdot\frac{n\pi d}{b}=\frac{4P}{ab}\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}$$

$$p(x,y)=\frac{4P}{ab}\sum_m\sum_n\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad \text{za} \quad m,n=1,2,3,... \quad ,$$

Primer 4



$u = c$

$v = \dfrac{b}{2}$

$d = \dfrac{b}{2}$

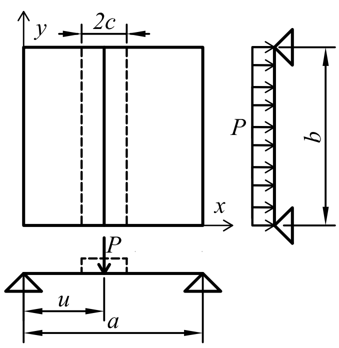
$$A_{mn} = \frac{4}{ab}\int_0^{2c}\int_0^{b} p\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \qquad \text{ili} \qquad A_{mn} = \frac{16p}{\pi^2 mn}\sin\frac{m\pi u}{a}\sin\frac{m\pi c}{a}\sin^2\frac{n\pi}{2}$$

$$A_{mn} = \begin{cases} \dfrac{16p}{\pi^2 mn}\sin^2\dfrac{m\pi c}{a}, m = 1,2,3,... \quad n = 1,3,5,... \\ 0, \qquad\qquad\qquad m = 1,2,3,... \quad n = 2,4,6,... \end{cases}$$

$$p(x,y) = \frac{16p}{\pi^2}\sum_m\sum_n \frac{\sin^2\dfrac{m\pi c}{a}}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \; m = 1,2,3,... \quad n = 1,3,5,...$$

Primer 5

Linijsko opterećenje prikazano na slici razviti u Furijeov red.
I u ovom slučaju linijsko opterećenje biće prikazano preko odgovarajućeg ravnomerno raspodeljenog opterećenja p koje deluje na površini $2c \cdot b$.



$$P = 2c \cdot p$$

$$p = \frac{P}{2c}$$

$$A_{mn} = \lim_{c\to 0} \frac{16p}{\pi^2 mn} \sin\frac{m\pi u}{a} \sin\frac{m\pi c}{a} = \lim_{c\to 0} \frac{8 \cdot 2p}{\pi^2 mn} \sin\frac{m\pi u}{a} \frac{\sin\frac{m\pi c}{a}}{\frac{m\pi c}{a}} \cdot \frac{m\pi c}{a}$$

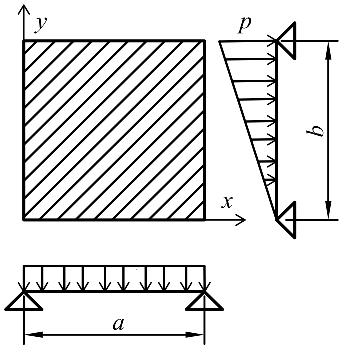
$$A_{mn} = \frac{8P}{a\pi n} \sin\frac{m\pi u}{a}$$

$$P(x,y) = \frac{8P}{\pi a} \sum_m \sum_n \frac{1}{n} \sin\frac{m\pi u}{a} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Primer 6

Opterećenje prikazano na slici razviti u Furijeov red.

$p(x,y)= \frac{p}{b} y$



$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b \frac{p}{b} y \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy = \frac{4p}{ab^2} \int_0^a \sin\frac{m\pi x}{a} dx \int_a^b y \sin\frac{n\pi y}{b} dy$$

Parcijalna integracija:

$$u = y \Rightarrow du = dy, \qquad dv = \sin\frac{n\pi y}{b} dy \Rightarrow v = -\frac{b}{n\pi}\cos\frac{n\pi y}{b}$$

$$\int_0^b y \sin\frac{n\pi y}{b} dy = -\frac{by}{n\pi}\cos\frac{n\pi y}{b}\Big|_0^b + \int_0^b \frac{b}{n\pi}\cos\frac{n\pi y}{b} dy =$$

$$= -\frac{b^2}{n\pi}\cos n\pi + \frac{b^2}{n^2\pi^2}\sin\frac{n\pi y}{b}\Big|_0^b = -\frac{b^2}{n\pi}\cos n\pi = -\frac{b^2}{n\pi}(-1)^n$$

$$A_{mn} = \frac{4p}{ab^2}\frac{(-a)}{m\pi}\cos\frac{m\pi x}{a}\Big|_0^a \left(\frac{-b^2}{n\pi}\right)(-1)^n =$$

$$A_{mn} = \frac{4p(-1)^n}{mn\pi^2}(\cos m\pi - 1) = \begin{cases} \frac{8p}{mn\pi^2}(-1)^{n+1}, & m = 1,3,5,\ldots \\ 0, & m = 2,4,6,\ldots \end{cases}$$

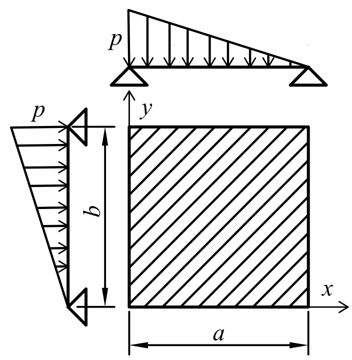
$$p(x,y)=\frac{8p}{\pi^2}\sum_m\sum_n\frac{(-1)^{n+1}}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\ m{=}2,4,6,...\quad n{=}1,2,3,...$$

Primer 7

Opterećenje prikazano na slici razviti u Furijeov red.

$$p(x)=\frac{p}{a}(a-x)$$

$$p(x,y)=\frac{p(x)}{b}\cdot y=\frac{p}{ab}(a-x)y$$



$$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b\frac{p}{ab}(a-x)y\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy=\frac{4p}{a^2b^2}\int_0^a(a-x)\sin\frac{m\pi x}{a}dx\int_0^b y\sin\frac{n\pi y}{b}dy=$$

$$=\frac{4p}{a^2b^2}\cdot\frac{a^2}{m\pi}\cdot\frac{b^2}{n\pi}(-1)^{n+1}=\frac{4p}{mn}(-1)^{n+1}$$

Parcijalna integracija:

$$u=a-x\qquad dv=\sin\frac{m\pi x}{a}dx$$

$$du=-dx\qquad v=-\frac{a}{m\pi}\cos\frac{m\pi x}{a}$$

$$\int_0^a(a-x)\sin\frac{m\pi x}{a}dx=-\frac{a(a-x)}{m\pi}\cos\frac{m\pi x}{a}\Bigg|_0^a-\frac{a}{m\pi}\int_0^a\cos\frac{m\pi x}{a}dx=\frac{a^2}{m\pi}$$

$$\int_0^b y\sin\frac{n\pi y}{b}dy=-\frac{b^2}{n\pi}(-1)^n=\frac{b^2}{n\pi}(-1)^{n+1}$$

$$p(x,y)=4p\sum_m\sum_n\frac{(-1)^{n+1}}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$